Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$-Optimal Control via Finite Dimensional Q-Parametrization and Linear Matrix Inequalities[1]

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Abstract

The problem of multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ optimal controller design is reviewed. There is as yet no exact solution to this problem. We present a method based on that proposed by Scherer [14]. The problem is formulated as a convex semidefinite program (SDP) using the LMI formulation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Suboptimal solutions are computed using finite dimensional Q-parametrization. The objective value of the suboptimal Q's converges to the true optimum as the dimension of Q is increased. State space representations are presented which are the analog of those given by Khargonekar and Rotea [11] for the $\mathcal{H}_2$ case. A simple example computed using FIR (Finite Impulse Response) Q's is presented.

1 Introduction

The multiobjective tradeoff paradigm [3] has become a very valuable design tool in engineering problems that have conflicting objectives. When the objectives being traded off are convex, very definitive conclusions can be obtained as to the feasibility or infeasibility of certain combinations of costs.

The multiobjective controller design problem has been solved exactly for the case where the tradeoff objectives are all $\mathcal{H}_2$ norms of various closed loop transfer functions [3, 11]. However, as soon as any other norm is introduced (eg: $\mathcal{H}_\infty$ or l_1), there is as yet no exact solution, and various approximations, relaxations, and bounds must be used [14, 11, 12, 16].

In [2], a pragmatic approach was taken: a finite dimensional Youla parameter Q was used, system impulse responses were truncated and infinite horizon costs and constraints were also truncated to a finite horizon. The problem was then solved as a standard constrained, convex optimization problem. One problem with this approach is that there is no guarantee that the controller designed in this way will be feasible in the true closed loop system, with respect to the infinite horizon costs and constraints. Another problem is that even when the dimension of Q is taken to be small, the method may produce very large optimization problems in situations where the systems in the Youla parametrization have very lightly damped modes. Despite these limitations, the method has produced some impressive designs.

In recent years, it has been shown that when a state space description is available, then many of the infinite horizon costs and constraints can be represented as linear matrix inequalities (LMIs) and minimized *exactly* and efficiently as semidefinite programs (SDPs), see [4, 13] for a catalog of such constraints. So in this paper, we take the design procedure of [2] to the natural next level and formulate the multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ problem using LMIs for the objectives and constraints and solve it as an SDP. In this way, the errors due to cost, constraint and pulse response truncation in [2] are eliminated.

We focus on the case where the objectives being traded off are all $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms of different discrete time closed loop transfer functions. This problem falls into the general class of problems that was thoroughly analyzed by Scherer [14]. There, the method we present in this paper was briefly mentioned and a very simple one-block SISO example was given. However, none of the details of the computational implementation were given. These turn out to be nontrivial in the MIMO case. We give an explicit description of the method, together with state space representations and complete statements of the semidefinite programs that must be solved for the general four-block MIMO case.

The LMI formulation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ costs [15, 4, 1, 7] introduces auxiliary Lyapunov matrices into the problem. As a result of product terms between these Lyapunov variables and the state space matrices of Q, the resulting constraints become nonlinear and hence, in general, nonconvex. In [13], convexity was recovered by a coordinate transformation of the controller variables, under the restriction that all the Lyapunov variables be equal. While this restriction makes the problem tractable in the LMI framework, it leads to conservatism in the overall design. There are as yet no results on the degree of this conservatism.

Other approaches to this problem include [12, 16, 5]. In [12], an exact solution to a heuristic upper bound is obtained. In [16, 5], the authors use the finite dimensional Q parametrization idea presented here to obtain sequences of problems whose optima converge to the true optimum. However, only the special case of the general problem treated here is considered: minimizing the $\mathcal{H}_2$ norm of a single closed loop (MIMO)

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transfer function subject to a single $\mathcal{H}_\infty$ constraint on another (MIMO) closed loop transfer is considered.

In the approach we present here, convexity in the LMI's is recovered by using an alternative state space description for the closed loop system, and by restricting the Youla parameter Q to a finite dimensional subspace of $\mathcal{H}_\infty$. The alternative state space description is obtained from the Youla parametrization via system Kronecker products. Similar techniques were used by Khargonekar and Rotea [11] for the multiobjective $\mathcal{H}_2$ case. The restriction on the dimension of Q also introduces conservatism in to the design. However, in contrast to [13], this conservatism can be made arbitrarily small by increasing the dimension of Q.

The only practical limitation on this approach is the size of the SDP's that can be solved. The method is demonstrated on a simple four-block problem.

2 Notation

Consider the general feedback system shown in Fig.1, where P is the open loop LTI plant $P: (w_e, u) \longrightarrow (z_r, y)$, K is the controller, and w_e, u, z_r, and y are the exogenous input, control input, regulated outputs, and sensed outputs, respectively. Let p and q be the dimensions of u and y, respectively; let m_i and n_i be the dimensions of z_i and w_i respectively; and let dimensions of the closed loop map G be $m \times n$.

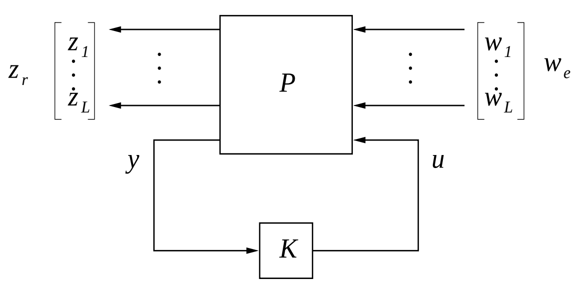

Fig.1 Multiobjective controller design problem.

The set of all achievable stable closed loop maps is given by:

$$\{G = P_{z_r w_e} + P_{z_r u} K (I - P_{yu} K)^{-1} P_{y w_e} \mid K \text{ stabilizing}\}$$

This representation is linear fractional, and a more convenient representation for us will be the equivalent Youla parametrization [3]:

$$\{G = H - U\,Q\,V \mid Q \in \mathcal{H}_\infty^{p\times q}\}$$

where Q is a free parameter in $\mathcal{H}_\infty^{p\times q}$ and the transfer matrices H, U, V and Q are all stable. This parametrization is affine in Q. Given state space realizations of H, U, Q and V, the closed loop transfer matrix G has the state space representation:

$$\left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] =$$
$$\left[\begin{array}{cccc|c} A_H & 0 & 0 & 0 & B_H \\ 0 & A_V & 0 & 0 & B_V \\ 0 & B_Q C_V & A_Q & 0 & B_Q D_V \\ 0 & B_U D_Q C_v & B_U C_Q & A_V & B_U D_Q D_V \\ \hline C_H & -D_U D_Q C_V & -D_U C_Q & -C_U & D_H - D_U D_Q D_V \end{array}\right]$$

where $A_H, \ldots, D_Q$ are the state space matrices of H, U, V and Q. From this, particular closed loop n_i-input/m_i-output transfer matrices G_i can be obtained as

$$G_i = L_i G R_i$$

where the matrices $L_i \in \mathbf{R}^{m_i \times m}$ and $R_i \in \mathbf{R}^{n \times n_i}$ select the appropriate channels [13]. The state space realization of G_i is then

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \left[\begin{array}{c|c} A_G & B_G R_i \\ \hline L_i C_G & L_i D_G R_i \end{array}\right]. \tag{1}$$

Similarly, we define H_i as

$$H_i = L_i H R_i$$

with the state space realization

$$\left[\begin{array}{c|c} A_{H_i} & B_{H_i} \\ \hline C_{H_i} & D_{H_i} \end{array}\right] = \left[\begin{array}{c|c} A_H & B_H R_i \\ \hline L_i C_H & L_i D_H R_i \end{array}\right].$$

In the next section, we will be concerned with the following vector valued cost $\mathcal{V}: \mathcal{H}^{p\times q} \to \mathbf{R}^L$:

$$\mathcal{V}(Q) := (\|G_1(Q)\|_2, \ldots, \|G_L(Q)\|_\infty). \tag{2}$$

where the first n_2 components of $\mathcal{V}$ are all $\mathcal{H}_2$-norms and last n_∞ components are all $\mathcal{H}_\infty$-norms, so $n_2 + n_\infty = L$. We will also use the notation $\|G_i\|_{\rho_i}$ for the components of $\mathcal{V}$, where

$$\rho_i = \begin{cases} 2 & ; i = 1, \ldots, n_2 \\ \infty & ; i = n_2 + 1, \ldots, L \end{cases}$$

3 Problem Statement

The notion of Pareto optimality defines what we mean by minimizing the vector valued cost $\mathcal{V}$. The following definition and theorem can be found for example in [11, 3].

Definition 1: *A $Q_{opt} \in \mathcal{H}_\infty^{p\times q}$ is* Pareto optimal *with respect to $\mathcal{V}$ iff there is no other $Q \in \mathcal{H}_\infty^{p\times q}$ such that $\|G_i(Q)\|_{\rho_i} \leq \|G_i(Q_{opt})\|_{\rho_i}$ for all i, and $\|G_{i_0}(Q)\|_{\rho_{i_0}} < \|G_{i_0}(Q_{opt})\|_{\rho_{i_0}}$ for at least one i_0.*

Theorem 1: *Define the set $\Lambda := \{\lambda \in \mathbf{R}^L : \lambda_i \geq 0, \sum_{i=1}^L \lambda_i = 1\}$ and define the* multiobjecitve cost *as $J_\lambda^M(Q) := \lambda^T \mathcal{V}(Q)$. Then the set of all Pareto optimal Q is the set of solutions (when they exist and when they are unique) of $\inf\{J_\lambda^M(Q) : Q \in \mathcal{H}_\infty^{p\times q}\}$ for all $\lambda \in \Lambda$.*

There is as yet no exact solution to the multiobjective problem of minimizing J_λ^M over $Q \in \mathcal{H}_\infty^{p\times q}$. The objective of this paper is to show how to compute Pareto optimal solutions of V using convex optimization.

Remark: For computational convenience, we redefine the vector valued cost $\mathcal{V}$ with all the entries squared, ie:

$$\mathcal{V}(Q) := (\|G_1(Q)\|_2^2, \ldots, \|G_L(Q)\|_\infty^2). \tag{3}$$

From definition 1 it is clear that the Pareto optimal points for this cost are the same as those of (2). Therefore Theorem 1 can still be used to generate the set of all Pareto optimal points.

4 Motivation

The method we present here works for multiobjective design with transfer matrices corresponding to *arbitrary* pairings of input and output vectors and norms. In this section, however, to motivate the problem, we will focus on the regulator problem, see Fig.2. The regulated output $z_r = (z, u)$ is made up of the regulated plant variables z and the control u; the exogenous input $w_e = (w, v)$ where w and v are process and sensor noises respectively.

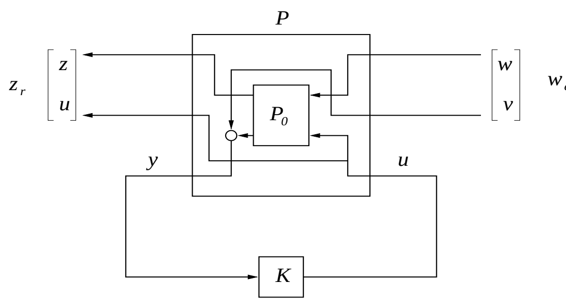

Fig.2 Regulator problem.

Typically we are interested in rejecting the disturbances w and v with efficient use of control u, eg: to avoid saturation. Thus the transfer matrices of interest are $G_1 = G_{zw_e}$ and $G_2 = G_{uw_e}$. In the case where very little is known about the disturbances w and v one might consider minimizing the response at z and u, due to the *worst case* input $w_e \in l_2$. This corresponds to minimizing $\|G_{zw_e}\|_\infty$ and $\|G_{uw_e}\|_\infty$ [9], simultaneously. Since these are usually *conflicting* requirements, one would like to do a tradeoff design as follows: The multiobjective $\mathcal{H}_\infty$ cost $J_\lambda^M(Q)$ in this case can be written as:

$$\begin{aligned} J_\lambda^M(Q) &= (1-\lambda)\,\|G_{zw_e}(Q)\|_\infty^2 + \lambda\,\|G_{uw_e}(Q)\|_\infty^2 \\ &= (1-\lambda)\sup_{w_e\neq 0}\frac{\|z\|_2^2}{\|w_e\|_2^2} + \lambda\sup_{w_e\neq 0}\frac{\|u\|_2^2}{\|w_e\|_2^2}. \end{aligned}$$

where $\lambda \in [0,1]$. Then carrying out the following:

for $\lambda \in [0,1]$
- solve the following for Q_λ: $\min_{Q\in\mathcal{H}_\infty} J_\lambda^M(Q)$
- plot $\|G_{zw_e}(Q_\lambda)\|_\infty$ versus $\|G_{uw_e}(Q_\lambda)\|_\infty$

end

generates the set of all Pareto optimal points Q_λ, and the tradeoff curve shown in Fig.3. This curve gives the set of all achievable pairs of values $(\|G_{uw_e}\|_\infty, \|G_{zw_e}\|_\infty)$, i.e., the *limits of performance* that are achievable with the given plant and the given cost function. This is very useful to know in practice.

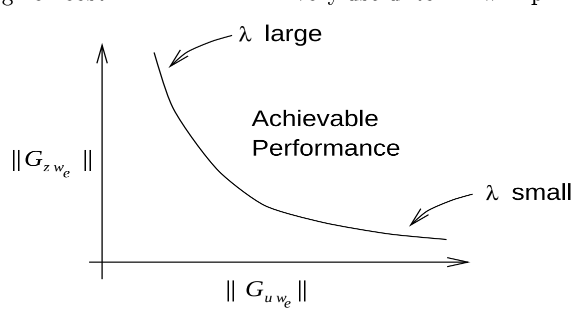

Fig.3 Tradeoff curve.

Note that minimizing J_λ^M is *not* a standard $\mathcal{H}_\infty$ problem. In the corresponding standard problem, one would set $\bar{z}_r = ((1-\lambda)^{1/2}z, \lambda^{1/2}u)$ and then minimize $\|G_{\bar{z}w_e}(Q)\|_\infty$, ie: minimize

$$\begin{aligned} J_\lambda^S(Q) &= \left\| \begin{bmatrix} (1-\lambda)^{1/2}G_{zw_e}(Q) \\ \lambda^{1/2}G_{uw_e}(Q) \end{bmatrix} \right\|_\infty^2 \\ &= \sup_{w_e\neq 0}\frac{((1-\lambda)\,\|z\|_2^2 + \lambda\,\|u\|_2^2)}{\|w_e\|_2^2} \end{aligned}$$

A fact which follows readily from the definitions of J_λ^M and J_λ^S is that:

$$\begin{aligned} & J_\lambda^S(Q) \le J_\lambda^M(Q) \quad \forall\, Q \in \mathcal{H}_\infty^{p\times q} \\ \Rightarrow & \inf_{Q\in\mathcal{H}_\infty^{p\times q}} J_\lambda^S(Q) \le \inf_{Q\in\mathcal{H}_\infty^{p\times q}} J_\lambda^M(Q). \end{aligned}$$

In the multiobjective design, the maximization of z and u over w_e is done *independently*, where as in the standard design, it is done *simultaneously*, which artificially couples z and u. However, in practice, the control effort and the regulated outputs are physically independent quantities, so why should we care about the *sum* of the gains at z and u? For example, they could peak at different frequencies.

Since $G_{zw_e}(Q)$ and $G_{uw_e}(Q)$ are affine in Q, both the multiobjective and standard problems are *convex* in Q. However, since Q is in $\mathcal{H}_\infty^{p\times q}$ (an infinite dimensional space) the problems of minimizing J_λ^M and J_λ^S are both infinite dimensional.

In the standard case of minimizing J_λ^S, state space structure provides means for converting the problem to a finite dimensional one which can then be minimized *exactly* via bisection and Riccati equations or via LMIs. Unfortunately, no such solutions are available for the multiobjective case. Broadly speaking, the approaches proposed in the literature obtain exact solutions to a heuristic upper bound [12], restrict the Lyapunov variables in the LMI's to be equal [13], or construct sequences of problems whose optima converge to the true optimum [16]. There is rarely any analysis given for the degree of conservatism, or the rate of convergence. Our approach falls into the third category.

5 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ LMI Constraints

We will now review a couple of standard general results on the representation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms as LMI constraints [4, 15, 1]. For convenience, they are stated in terms of the notation for the G_i's.

Lemma 1: ($\mathcal{H}_2$ norm bound) *Given any transfer function* $G_i \equiv D_i + C_i(zI - A_i)^{-1}B_i$ *(not necessarily minimal), we have:*

$$\|G_i\|_2^2 \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{Tr}\left(G_i(e^{j\omega})^* G_i(e^{j\omega})\right) d\omega < \beta_i^2$$

$$A_i \;\; \textit{asymptotically stable}$$

if and only if the following LMI in X_i *and* S_i *is feasible:*

$$\begin{gathered} \begin{bmatrix} A_i^T X_i A_i - X_i & A_i^T X_i B_i \\ B_i^T X_i A_i & B_i^T X_i B_i - I \end{bmatrix} < 0 \\ \begin{bmatrix} X_i & 0 & C_i^T \\ 0 & I & D_i^T \\ C_i & D_i & S_i \end{bmatrix} > 0 \\ \mathbf{Tr}(S_i) - \beta_i^2 < 0 \\ X_i > 0. \end{gathered} \qquad (4)$$

Lemma 2: ($\mathcal{H}_\infty$ norm bound) *Given any transfer function $G_i \equiv D_i + C_i\,(zI - A_i)^{-1} B_i$ (not necessarily minimal), we have:*

$$\|G_i\|_\infty < \gamma_i$$
$$A_i \ \textit{asymptotically stable}$$

if and only if the following LMI in X_i is feasible:

$$\begin{bmatrix} A_i^T X_i A_i - X_i & A_i^T X_i B_i & C_i^T \\ B_i^T X_i A_i & B_i^T X_i B_i - \gamma_i I & D_i^T \\ C_i & D_i & -\gamma_i I \end{bmatrix} < 0 \qquad X_i > 0. \tag{5}$$

This latter lemma is known as the Bounded Real lemma. Note that in both lemmas, the (1,1)-block in the first inequality must be negative. Thus it follows from Lyapunov theory that when A_i is known to be stable, the $X_i > 0$ constraint is automatically satisfied by any feasible X_i. In this case it can be dropped.

Now consider lemma 1. As it stands, the lemma allows us to check if the $\mathcal{H}_2$ norm of G_i is bounded by some β_i. However, the LMI is in fact *jointly* linear in the variables $(\beta_i^2, S_i, X_i, C_i, D_i)$. Thus if A_i and B_i were fixed, then one could solve the following optimization problem in the variables $(\beta_i^2, S_i, X_i, C_i, D_i)$ *simultaneously*, and thus compute C_i and D_i which minimize $\|G_i\|_2$:

$$\begin{array}{ll} \text{minimize} & \beta_i^2 \\ \text{subject to} & (4). \end{array} \tag{6}$$

Similarly for the $\mathcal{H}_\infty$ case, if A_i and B_i were fixed, one could compute the C_i and D_i which minimize $\|G_i\|_\infty$ by solving:

$$\begin{array}{ll} \text{minimize} & \gamma_i^2 \\ \text{subject to} & (5). \end{array} \tag{7}$$

Both of these problems can be easily cast as SDP's.

However, in the realization of G_i given in (1), the matrices of Q appear not only in C_i and D_i, but also in A_i and B_i. This makes the problem nonconvex due to the cross terms between X_i and A_i and B_i. To recover convexity, we will use a finite dimensional basis for Q and an alternative state space realization for G_i to keep all the variable parameters of Q in C_i and D_i.

6 State Space SISO FIR

In what follows, we will use a MIMO N-tap FIR Q-parameter. Let Q_{rs} be the individual SISO FIR component systems in Q with pulse response $\{q_{rs}(0), q_{rs}(1), \ldots, q_{rs}(N-1), 0, 0, \ldots\}$. These can be realized as:

$$\left[\begin{array}{c|c} A_{Q_{rs}} & B_{Q_{rs}} \\ \hline C_{Q_{rs}} & D_{Q_{rs}} \end{array}\right] = \left[\begin{array}{c|c} Z & e_1 \\ \hline q_{rs} & q_{0,rs} \end{array}\right]$$

where $Z \in \mathbf{R}^{(N-1)\times(N-1)}$ is the "shift matrix" made up of all zeros except for ones on the first subdiagonal; e_1 is the first column of $I_{(N-1)}$, $q_{rs} = [q_{rs}(1) \ldots q_{rs}(N-1)]$ (row matrix), and $q_{0,rs} = q_{rs}(0)$. We note that all coefficients of Q_{rs} appear only in $C_{Q_{rs}}$ and $D_{Q_{rs}}$.

Remark: Setting the Q_{rs} to be FIR's corresponds to choosing the basis $\{z^0, z^{-1}, \ldots, z^{-(N-1)}\}$. We note, however, that *any* other basis could be used by simply by specifying different $A_{Q_{rs}}$ and $B_{Q_{rs}}$ matrices.

7 Alternative State Space Representation for G_i

The next step towards recovering convexity in the LMI's comes from the following simple lemma:

Lemma 3: *The transfer functions G_i can be written as*

$$G_i(Q) = H_i - \sum_{r,s} Q_{rs} \otimes T_{rs,i},$$

where the Q_{rs} are the individual SISO entries of Q, $T_{rs,i} := (L_i U e_r)(e_s^T V R_i)$, e_i is the ith column of the identity matrix of appropriate dimensions, and $Q_{rs} \otimes T_{rs,i}$ denotes the system Kronecker product [11] in $\mathcal{H}_\infty$ of Q_{rs} with $T_{rs,i}$, defined as:

$$Q_{rs} \otimes T_{rs,i} := \begin{bmatrix} Q_{rs}\, T_{rs,i}^{(11)} & \cdots & Q_{rs}\, T_{rs,i}^{(1n)} \\ \vdots & \ddots & \vdots \\ Q_{rs}\, T_{rs,i}^{(m1)} & \cdots & Q_{rs}\, T_{rs,i}^{(mn)} \end{bmatrix}.$$

Proof: First, recall that in the Q-parametrization: H, U, Q, and V are simply matrices in $\mathcal{H}_\infty$. Decomposing Q as the sum of its SISO components Q_{rs} multiplied by the elementary matrices $E_{rs} = e_r\, e_s^T$ gives:

$$\begin{aligned} G_i(Q) &= H_i - L_i U (\sum_{r,s} Q_{rs} e_r e_s^T) V R_i \\ &= H_i - \sum_{r,s} Q_{rs} \left((L_i U e_r)(e_s^T V R_i) \right) \\ &= H_i - \sum_{r,s} Q_{rs}\, T_{rs,i} \end{aligned}$$

where $T_{rs,i} = (L_i U e_r)(e_s^T V R_i)$. Now each product term $Q_{rs}\, T_{rs,i}$ is just a scalar (SISO) times a matrix (MIMO) in $\mathcal{H}_\infty$. So in fact: $Q_{rs}\, T_{rs,i} = Q_{rs} \otimes T_{rs,i}$ by the definition of the multiplication of a matrix by a scalar in $\mathcal{H}_\infty$. ∎

Note that $T_{rs,i}$ is just the series connection of the systems $L_i U e_r$ and $e_s^T V R_i$, so its state space realization is:

$$\begin{aligned} &\left[\begin{array}{c|c} A_{T_{rs,i}} & B_{T_{rs,i}} \\ \hline C_{T_{rs,i}} & D_{T_{rs,i}} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_V & 0 & B_V R_i \\ B_U e_r e_s^T C_V & A_U & B_U e_r e_s^T D_V R_i \\ \hline L_i D_U e_r e_s^T C_V & L_i C_U & L_i D_U e_r e_s^T D_V R_i \end{array}\right]. \end{aligned}$$

Next, using a formula from [11], we obtain a (nonunique) state space realization of the system Kronecker product $Q_{rs} \otimes T_{rs,i}$ in terms of state space matrices of Q_{rs} and $T_{rs,i}$:

$$\begin{aligned} &\left[\begin{array}{c|c} A_{rs,i} & B_{rs,i} \\ \hline C_{rs,i} & D_{rs,i} \end{array}\right] = \\ &\left[\begin{array}{cc|c} A_{Q_{rs}} \otimes I_{m_i} & B_{Q_{rs}} \otimes C_{T_{rs,i}} & B_{Q_{rs}} \otimes D_{T_{rs,i}} \\ 0 & A_{T_{rs,i}} & B_{T_{rs,i}} \\ \hline C_{Q_{rs}} \otimes I_{m_i} & D_{Q_{rs}} \otimes C_{T_{rs,i}} & D_{Q_{rs}} \otimes D_{T_{rs,i}} \end{array}\right]. \end{aligned}$$

where m_i is the number of outputs of G_i.

Finally, we arrive at the desired alternative state space representation for G_i: just add H_i in parallel with all the systems $Q_{rs} \otimes T_{rs,i}$:

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \tag{8}$$

$$\left[\begin{array}{cccc|c} A_{H_i} & & & & B_{H_i} \\ & A_{11,i} & & & B_{11,i} \\ & & \ddots & & \vdots \\ & & & A_{pq,i} & B_{pq,i} \\ \hline C_{H_i} & -C_{11,i} & \cdots & -C_{pq,i} & D_{H_i} - \sum_{r,s} D_{rs,i} \end{array}\right].$$

From the above, we see that if the Q_{rs} have the SISO FIR structure $\{Z, e_1, q_{rs}, q_{0,rs}\}$ then all the coefficients of Q_{rs} will appear only in the C and D matrices of $Q_{rs} \otimes T_{rs,i}$. Specifically, we have:

$$\begin{array}{rcl} C_{rs,i} & = & [q_{rs} \otimes I_{m_i} \quad q_{0,rs} \otimes C_{T_{rs,i}}] \\ D_{rs,i} & = & [q_{0,rs} \otimes D_{T_{rs,i}}] \end{array} \tag{9}$$

which are both *linear* in the coefficients of Q_{rs}. Hence all the coefficients of Q in (8) appear (linearly) only in C_i and D_i, which is what we wanted. We emphasize this fact by writing C_i and D_i as $C_i(\mathbf{q})$ and $D_i(\mathbf{q})$, where $\mathbf{q}$ is the concatenation of all the coefficients of Q, ie:

$$\mathbf{q} := [[q_{0,11}, q_{11}] \cdots [q_{0,pq}, q_{pq}]]^T .$$

Furthermore, A_i and B_i are then fixed. Also, A_i is *stable* since A_{H_i}, A_U, Z, and A_V are all stable.

8 Solution of Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

Let us define f_i and g_i as the constraints obtained from applying the LMI conditions (4) and (5) to the alternative state space realization (8) and (9), i.e.

$$f_i(\beta_i, S_i, X_i, \mathbf{q}) := \mathbf{diag}\left(\begin{bmatrix} A_i^T X_i A_i - X_i & A_i^T X_i B_i \\ B_i^T X_i A_i & B_i^T X_i B_i - I \end{bmatrix}, - \begin{bmatrix} X_i & 0 & C_i(\mathbf{q})^T \\ 0 & I & D_i(\mathbf{q})^T \\ C_i(\mathbf{q}) & D_i(\mathbf{q}) & S_i \end{bmatrix}, \mathbf{Tr}(S_i) - \beta_i^2 \right),$$

$$g_i(\gamma_i, X_i, \mathbf{q}) := \begin{bmatrix} A_i^T X_i A_i - X_i & A_i^T X_i B_i & C_i(\mathbf{q})^T \\ B_i^T X_i A_i & B_i^T X_i B_i - \gamma_i I & D_i(\mathbf{q})^T \\ C_i(\mathbf{q}) & D_i(\mathbf{q}) & -\gamma_i I \end{bmatrix},$$

The $X_i > 0$ constraint has been dropped due to the stability of A_i.

From the formulations in (6) and (7), it follows that $\|G_i\|_2$ can be minimized by solving the following optimization problem in the variables $(\beta_i, S_i, X_i, \mathbf{q})$:

$$\begin{array}{ll} \text{minimize} & \beta_i^2 \\ \text{subject to} & f_i(\beta_i, S_i, X_i, \mathbf{q}) < 0 \end{array}$$

and $\|G_i\|_\infty$ can be minimized by solving the SDP in the variables $(\gamma_i, X_i, \mathbf{q})$:

$$\begin{array}{ll} \text{minimize} & \gamma_i^2 \\ \text{subject to} & g_i(\gamma_i, X_i, \mathbf{q}) < 0 \end{array} .$$

Again, both of these problems can easily be solved as SDP's. Furthermore, using arguments similar to those in [14], it can be shown that as the number of taps goes to infinity, the objective values of the Q's which optimize the SDP's above will converge to the true optimum from above.

Actually, in [7] a very similar approach was used to solve the standard $\mathcal{H}_\infty$ problem *exactly*, without using finite dimensional approximations of Q or the alternative state space realization. Unfortunately, that approach cannot be extended to handle multiobjective problems without making some assumptions which lead to conservatism in the design [13].

On the other hand, the approach presented here extends *trivially* to the multiobjective case: $J_\lambda^M(Q) = \sum_{i=1}^{n_2} \lambda_i \|G_i(Q)\|_2^2 + \sum_{i=n_2+1}^{L} \lambda_i \|G_i(Q)\|_\infty^2$ can be minimized by solving the following optimization problem in the variables $(\beta_1, S_1, X_1, \ldots, \gamma_L, X_L, \mathbf{q})$:

$$\begin{array}{lll} \min & \sum_{i=1}^{n_2} \lambda_i \beta_i^2 + \sum_{i=n_2+1}^{L} \lambda_i \gamma_i^2 & \\ \text{s.t.} & f_i(\beta_i, S_i, X_i, \mathbf{q}) < 0 & i = 1, \ldots, n_2 \\ & g_i(\gamma_i, X_i, \mathbf{q}) < 0 & i = n_2 + 1, \ldots, L \end{array}$$

Of course, this can also be solved as an SDP.

9 Numerical Example

As an example, we now consider the four-block problem of stabilizing an unstable continuous time second order system, with a discrete-time controller. We assume unknown discrete-time process and measurement noises as shown in Fig.2. Such problems could arise, for example, in the stabilization of charged particle beams in circular accelerators [10]. The continuous time model (in normalized units) is:

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -\omega_0^2 & -2\,\zeta\,\omega_0 & \alpha \\ \hline 1 & 0 & 0 \end{array}\right],$$

where $\omega_0 = 2\pi$, $\zeta = -0.5$, $\alpha = 4\omega_0^2$. The system was zero-order-hold discretized at a sampling period of $T_s = 0.17$s, with a $0.9T_s$ delay in the feedback loop path, to give the final discrete-time plant P_0:

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{ccc|c|c} 0.2726 & 0.2443 & 0.7217 & 0.7274 & 0.0057 \\ -9.6460 & 1.8078 & 8.9537 & 9.6460 & 0.6924 \\ 0 & 0 & 0 & 0 & 1.0000 \\ \hline 4 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right],$$

(The sensor readout matrix C_2 has been made factor of 4 smaller than C_1.) Then P was obtained from P_0 as shown in Fig.2.

The solid tradeoff curve in Fig.4 was obtained by minimizing the multiobjective cost J_λ^M for $\lambda \in [0,1]$, as described in section 3. For comparison, the dashed curve was obtained by solving the standard problem of minimizing J_λ^S. An LQG controller was used to obtain H, U, and V (see [6] p.107) and a 12-Tap FIR Q parameter was used.

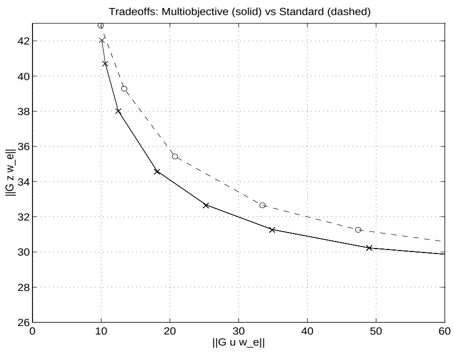

Fig.4 Multiobjective Tradeoff curve.

These curves show that there can be a significant difference in performance, between standard (dashed) and multiobjective (solid) controllers. Over most of the curve, the multiobjective controller offers the same regulation of state excursions, for upto 25% less actuator effort. In fact, we can expect this kind of behavior in general: since the multiobjective design paradigm generatees Pareto optimal points, we are guaranteed that the multiobjective curve will always minorize the corresponding curves of any other controllers. This is provided, of course, that enough taps are used in Q.

10 Conclusion

The problem of multiobjective optimal controller tradeoff design when the objectives are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, as yet, cannot be solved exactly. Existing methods are either approximate or conservative. We have presented an approximate solution based on that proposed in [14]. Using standard LMI representations of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, and a finite dimensional Youla parameter Q, a finite dimensional approximation to the original infinite dimensional problem was obtained. As the dimension of Q is increased, the objective values of the optimal Q converge to the true optimum from above [14].

By using an alternative state space realization for the closed loop system, the finite dimensional problem was reduced to a semidefinite program which is convex in the variables of the finite dimensional Q. Explicit state space matrices for all the relevant transfer matrices were obtained using system Kronecker products, similar to those given in [11] for the pure $\mathcal{H}_2$ multiobjective problem. The method is easy to implement in practice on any SDP solver eg: [8, 17, 18].

Our approach offers the following advantages over existing methods: First, as a benefit of using state space matrices rather than impulse responses, all errors due to the truncation of infinite horizon costs and constraints in [2] are eliminated. The controller is thus guaranteed to be feasible for the true system with the true costs. Second, in contrast to the method in [13], since we do not restrict the Lyapunov matrices to be all equal, the conservatism in our method can, in principle, be made arbitrarily small by choosing an appropriate basis of sufficient dimension.

There are two main drawbacks to our approach: First, as with most finite dimensional Q based approaches, we have no analysis on the rate of convergence of the SDP optima to the true optimum in infinite dimensions, or on their degree of suboptimality. Second, the exactness of the LMI formulation is at the cost of introducing a large number of auxiliary variables, namely the Lyapunov matrices X_i. This produces an order of magnitude increase in the number of variables which is actually compounded by the sparse alternative state space realization that was used to recover convexity. This can be a problem when the system or the number of basis functions is large.

In closing, we would like to point out that the approach outlined here is not necessarily limited to just $\mathcal{H}_2/\mathcal{H}_\infty$ multiobjective design. It may be possible to apply the same idea to other standard LMI constraints, see [13]. Also, an entirely parallel development should be possible for continuous-time systems [3, 4, 5, 14].

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